The Structure of Weyl Alternation Sets

Partial Orders and Independence Systems

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(based on joint work with Portia X. Anderson, Esther Banaian, Melanie J. Ferreri, Owen C. Goff, Kimberly P. Hadaway, Pamela E. Harris, Kimberly J. Harry, Nicholas Mayers, Shiyun Wang)

Overview

1. **What (and Why) is a Weyl Alternation Set?**
2. **Poset Structure**
3. **Independence System Structure**
4. **Examples: $\mathcal{A}(\tilde{\alpha}, \mu)$ in Type A**
5. **Enumeration and Recurrences in Type A**

What (and Why) is a Weyl Alternation Set?

Lie Algebras

A **Lie algebra** $\mathfrak{g}$ is a vector space over $\mathbb{C}$ equipped with an operation called a Lie bracket.

$\mathfrak{sl}_n$

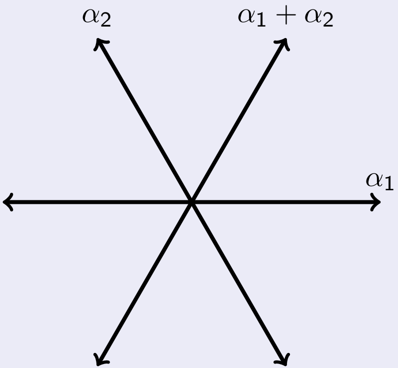
The Lie algebra $\mathfrak{sl}_n$ consists of $(n+1) \times (n+1)$ matrices over $\mathbb{C}$ with trace zero and Lie bracket

$$[X, Y] = XY - YX.$$

Roots

A simple Lie algebra $\mathfrak{g}$ has an associated irreducible **root system** Φ, which we'll think of as vectors in Euclidean space.

$\mathfrak{sl}_2$



- Simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$
- Positive Roots Φ^+
- Negative Roots $\Phi^- = -\Phi^+$

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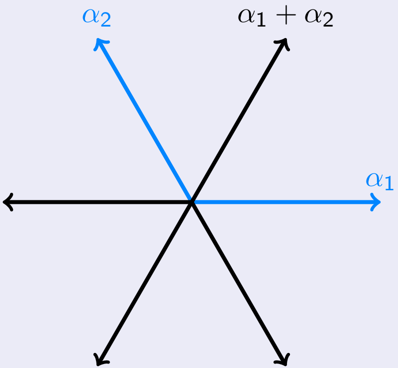

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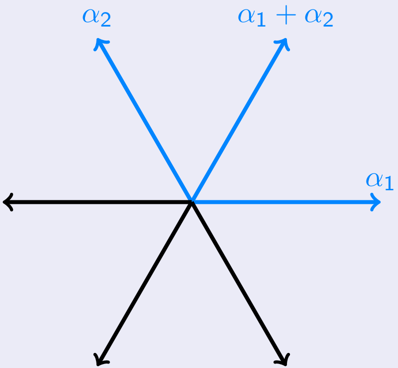

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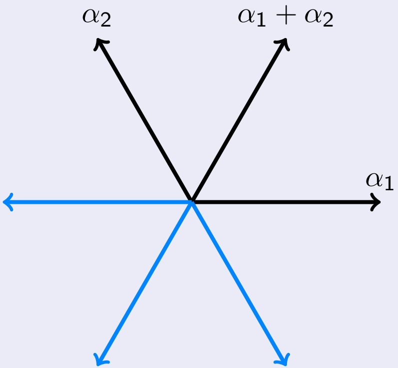

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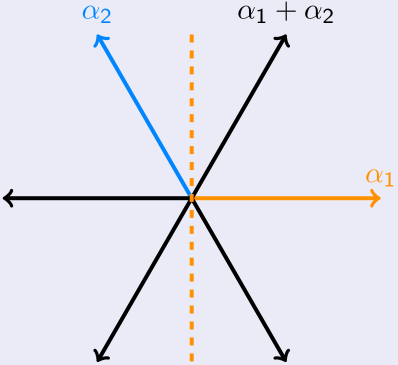
Weyl Group

For a root system with simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$, the corresponding **Weyl group** W is generated by reflections $s_1, \dots, s_r$ where s_i is the reflection through the hyperplane orthogonal to α_i.

$\mathfrak{sl}_2$



$$s_1(\alpha_2) = \alpha_1 + \alpha_2$$

Weights

- A **representation** V of $\mathfrak{g}$ is a map $\mathfrak{g} \to \mathfrak{gl}(V)$ respecting the Lie bracket.
- A **weight space** is a generalized eigenspace. Formally, if $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra, then a **weight** λ is a linear functional $\lambda : \mathfrak{h} \to \mathbb{C}$, and the corresponding weight space is
$$V_\lambda = \{v \in V | \forall H \in \mathfrak{h}, Hv = \lambda(H)v\}.$$
- A simple $\mathfrak{g}$-representation is determined by its highest weight. For V the representation with highest weight λ, we we write
$$m(\lambda, \mu) = \dim(V_\mu)$$
for the **multiplicity** of μ in V.

Weights

We can think of weights as living in Euclidean space along with roots *(roots are weights of the adjoint representation)*.

- Write (λ, α) for the inner product in this Euclidean space
- The Weyl group acts as $s_i(\lambda) = \lambda - 2\frac{(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$

Property	**Definition**
λ dominant	$(\lambda, \alpha) \geq 0$ for all $\alpha \in \Phi^+$
λ integral	$2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$
$\lambda \leq \mu$	$\mu - \lambda$ can be written as a positive linear combination of positive roots

We say that μ is **higher** than λ whenever $\lambda < \mu$.

Kostant's Weight Multiplicity Formula

Theorem (Kostant 1958)

The multiplicity of the weight μ in the representation V of $\mathfrak{g}$ with highest weight λ is

$$m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - \mu - \rho)$$

where

- $\ell(\sigma)$ *is the minimum number of reflections needed to write* σ*,*
- $\wp(\xi)$, *the* **Kostant partition function**, *is the number of ways to write* ξ *as a non-negative integer linear combination of positive roots* Φ^+, *and*
- $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

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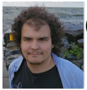
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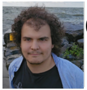

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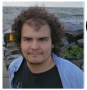

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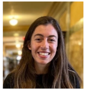


Esther Banaian

★☆☆☆☆

Tried to use it for the type B8 Lie algebra, but it told me I had to sum over more than 10 million terms! What gives??

Weyl Alternation Set

For some elements $\sigma \in W$, we have that $\wp(\sigma(\lambda+\rho)-\mu-\rho)=0$, so they don't contribute to the sum. The **Weyl alternation set** is the set of elements that *do* contribute:

$$\mathcal{A}(\lambda,\mu)=\{\sigma \in W : \wp(\sigma(\lambda+\rho)-\mu-\rho)>0\}$$

Note that $\sigma \in \mathcal{A}(\lambda,\mu)$ if and only if $\sigma(\lambda+\rho)-\mu-\rho$ is a linear combination of positive roots with nonnegative (not all zero) coefficients.

We can take the sum in our formula over only elements of $\mathcal{A}(\lambda,\mu)$ instead of the full Weyl group.

Poset Structure

Weak Order

A **reduced expression** of an element $\sigma \in W$ is a minimum length expression for σ as a product of simple transpositions s_i.

The **left weak order** $(W, \leq_L)$ is defined by $\sigma \leq_L \tau$ if a reduced expression for σ is a suffix of a reduced expression for τ.

Example

$$s_1 s_3 \leq_L s_1 s_2 s_1 s_3$$

The **right weak order** $(W, \leq_R)$ is defined by $\sigma \leq_R \tau$ if a reduced expression for σ is a prefix of a reduced expression for τ.

Example

$$s_1 s_2 \leq_R s_1 s_2 s_1 s_3$$

Poset Structure of $\mathcal{A}(\lambda, \mu)$

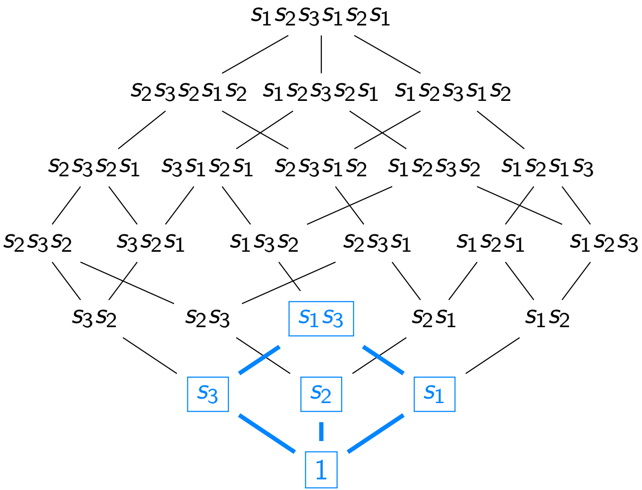
Theorem

Let λ be an integral dominant weight of a simple Lie algebra $\mathfrak{g}$ with Weyl group W. Then for any weight μ, the Weyl alternation set $\mathcal{A}(\lambda, \mu)$ is a (possibly empty) order ideal in the left and right weak orders of W.

Corollary

If $\sigma \in \mathcal{A}(\lambda, \mu)$, then any contiguous subword of a reduced expression for σ is also in $\mathcal{A}(\lambda, \mu)$.

Poset Structure of $\mathcal{A}(\lambda, \mu)$



The left weak order on the type A_3 Weyl group with the set

$$\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$$

highlighted where

$$\tilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3.$$

Forbidden Words

Corollary

If $\sigma \in \mathcal{A}(\lambda, \mu)$, then any contiguous subword of a reduced expression for σ is also in $\mathcal{A}(\lambda, \mu)$.

Kimberly P. Hadaway

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Here's a clever hack: the contrapositive of this theorem doubles as a way to prove elements *aren't* in the alternation set!

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Contrapositive

If $\sigma \notin \mathcal{A}(\lambda, \mu)$, then any word containing a reduced expression of σ as a contiguous subword is also not in $\mathcal{A}(\lambda, \mu)$.

Independence System Structure

Prior Work

Notation: $\tilde{\alpha}$ is the **highest root**. In type A_r, it's just $\tilde{\alpha} = \alpha_1 + \alpha_2 + \cdots + \alpha_r$.

Theorem (Harris 2011)

In type A_r, the Weyl alternation set $\mathcal{A}(\tilde{\alpha}, 0)$ consists of commuting products of simple transpositions s_i for $1 < i < r$.

Theorem (Harry 2024)

In type A_r, the Weyl alternation set $\mathcal{A}(\tilde{\alpha}, \mu)$ for $\mu = \alpha_k + \alpha_{k+1} + \cdots + \alpha_\ell$ a positive root consists of commuting products of simple transpositions s_i for $1 < i < k$ or $\ell < i < r$.

These theorems describe the alternation set like an **independence system** (where a set of simple transpositions is independent if they commute pairwise).

Independence Systems

Independence System (a.k.a Abstract Simplicial Complex)

An independence system is a pair $(V, \mathcal{I})$ consisting of a finite set V and a collection of subsets of $\mathcal{I}$ called **independent sets** satisfying

1. $\emptyset \in \mathcal{I}$
2. If $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$.

To extend the results of Harris and Harry, we need to generalize their notion of independence (commuting).

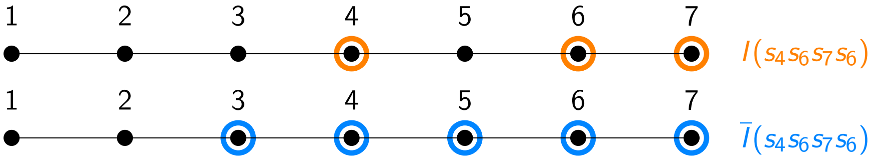
Influence

Influence

For an element $\sigma \in W$ of the Weyl group, define the **influence** and **extended influence** of σ, denoted $I(\sigma)$ and $\bar{I}(\sigma)$ respectively, as follows:

$$I(\sigma) = \{i : s_i \text{ is in a reduced word for } \sigma\}$$
$$\bar{I}(\sigma) = \{i : i \in I(\sigma) \text{ or } i \text{ is adjacent to some } j \in I(\sigma)\}$$

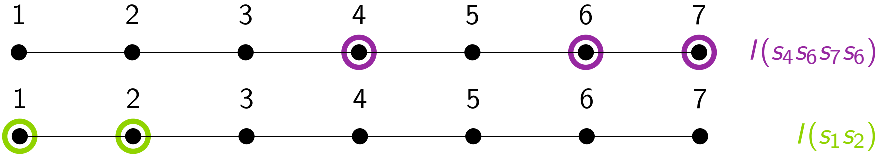
("adjacent" meaning "adjacent in the Dynkin diagram")



Independence

Call a subset $X \subseteq W$ **independent** if for each $\sigma, \tau \in X$ with $\sigma \neq \tau$, we have that

$$I(\sigma) \cap \overline{I}(\tau) = \emptyset.$$



Basic Allowable Subwords

Theorem

Let λ be a dominant integral weight of a simple Lie algebra $\mathfrak{g}$ and μ a weight such that $\mathcal{A}(\lambda, \mu)$ is nonempty.

1. *There exists a unique subset $S \subseteq \mathcal{A}(\lambda, \mu)$ with $1 \notin S$ such that each $b \in S$ has connected influence and any element $\sigma \in \mathcal{A}(\lambda, \mu)$ can be written as a product of an independent subset of S.*

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2. *Furthermore, there is a bijection between elements of $\mathcal{A}(\lambda, \mu)$ and independent subsets of S where each independent subset corresponds to its product.*

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2. *Furthermore, there is a bijection between elements of $\mathcal{A}(\lambda, \mu)$ and independent subsets of S where each independent subset corresponds to its product.*

Call this unique subset $\mathrm{BAS}(\lambda, \mu)$, the **basic allowable subwords**.

Summary

Given weights λ (dominant, integral) and μ, $\mathcal{A}(\lambda, \mu)$ has the structure of an independence system $(\mathrm{BAS}(\lambda, \mu), \mathcal{I})$. We can rephrase the theorems of Harris and Harry in this language.

Theorem (Harris 2011)

In type A_r,

$$\mathrm{BAS}(\tilde{\alpha}, 0) = \{s_i : 1 < i < r\}.$$

Theorem (Harry 2024)

In type A_r,

$$\mathrm{BAS}(\tilde{\alpha}, \alpha_k + \cdots + \alpha_\ell) = \{s_i : 1 < i < k \text{ or } \ell < i < r\}.$$

Examples: $\mathcal{A}(\tilde{\alpha}, \mu)$ in Type A

The Case of $\mu = -\tilde{\alpha}$

Theorem

The set $\mathrm{BAS}(\tilde{\alpha}, -\tilde{\alpha})$ *of basic allowable subwords of* $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$ *in type* A_r *consists of*

(a) s_k *with* $1 \leq k \leq r$,

(b) $s_{k+1}s_k$ *with* $2 \leq k \leq r-2$,

(c) $s_k s_{k+1}$ *with* $2 \leq k \leq r-2$,

(d) $s_k s_{k+1} s_k$ *with* $2 \leq k \leq r-2$, *and*

(e) $s_{k+2} s_k s_{k+1}$ *with* $2 \leq k \leq r-3$.

If we wanted to build an element of $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$ in type A_9, we could take:

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$s_2 s_3$

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If we wanted to build an element of $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$ in type A_9, we could take:

$$s_2 s_3 \;\; s_5 s_6 s_5 \left\{ \begin{array}{c} s_7 s_6 \\ s_4 \\ s_8 s_9 \end{array} \right\}$$

The Case of $\mu = -\tilde{\alpha}$

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If we wanted to build an element of $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$ in type A_9, we could take:

$$s_2 s_3 \; s_5 s_6 s_5 \left\{ \begin{matrix} s_7 s_6 \\ s_4 \\ \mathbf{s_8 s_9} \end{matrix} \right\}$$

A Useful Lemma

Lemma

Let $S \subseteq \mathcal{A}(\lambda, \mu)$ contain all simple transpositions in $\mathcal{A}(\lambda, \mu)$ but not the identity. Suppose that for any $\sigma, \tau \in S$ non-independent elements the product $\sigma\tau$ falls into one of the following 3 cases:

1. $\sigma\tau \in S$,
2. $\sigma\tau = \nu_1\nu_2 \cdots \nu_m$ *where* $\{\nu_1, \nu_2, \ldots, \nu_m\}$ *is a (possibly empty) independent subset of* S *and* $\ell(\nu_1) + \cdots + \ell(\nu_m) < \ell(\sigma) + \ell(\tau)$, *or*
3. $\sigma\tau$ *contains a forbidden subword (i.e. a contiguous substring not in* $\mathcal{A}(\lambda, \mu)$*).*

Then $\mathrm{BAS}(\lambda, \mu) = S$.

Proof Method

We establish a set of words not contained in $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$.

Forbidden Words

The following strings cannot appear in any σ in $\mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$.

1. $s_2 s_1, s_1 s_2, s_{r-1} s_r, s_r s_{r-1}$
2. $s_{i-1} s_i s_{i+1}, s_i s_{i-1} s_{i+1}, s_{i+1} s_i s_{i-1}$
3. The product of four consecutive s_i in any order

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2. $s_{i-1} s_i s_{i+1}, s_i s_{i-1} s_{i+1}, s_{i+1} s_i s_{i-1}$
3. The product of four consecutive s_i in any order

For any pair of non-independent elements of S, we check that its product falls into a case in the previous lemma.

Examples

- $(s_{k+1} s_k)(s_k s_{k+1} s_k) = s_k \in S$
- $(s_{k+1} s_k)(s_k s_{k-1}) = s_{k+1} s_{k-1}$ is an independent product of elements in S
- $(s_{k+1} s_k)(s_{k+1} s_{k+2})$ contains $s_k s_{k+1} s_{k+2} \notin \mathcal{A}(\tilde{\alpha}, -\tilde{\alpha})$

Moving Between Roots

Theorem

Let λ be an integral dominant weight, and let μ and ν be two other integral weights such that $\nu \leq \mu$. Then, $\mathrm{BAS}(\lambda, \mu) = \mathrm{BAS}(\lambda, \nu) \cap \mathcal{A}(\lambda, \mu)$.

Idea: If we know the basic allowable subwords for a root ν, we can quickly compute the basic allowable subwords for a higher root μ.

The Case of $\mathcal{A}(\tilde{\alpha}, \mu)$ for μ any root in Type A_r

We use the notation $\alpha_{i,j} = \sum_{k=i}^{j} \alpha_k$. Each table entry is a range of possible values for k.

μ	s_k	$s_{k+1}s_k$	$s_k s_{k+1}$	$s_k s_{k+1} s_k$	$s_k s_{k+2} s_{k+1}$
$-\tilde{\alpha}$	$[1, r]$	$[2, r-2]$	$[2, r-2]$	$[2, r-2]$	$[2, r-3]$
$-\alpha_{1,j}$	$[1, r-1]$	$[2, j-1]$	$[2, j^*]$	$[2, j-1]$	$[2, j-2]$
$-\alpha_{i,r}$	$[2, r]$	$[i^*-1, r-2]$	$[i, r-2]$	$[i, r-2]$	$[i, r-3]$
$-\alpha_{i,j}$	$[2, r-1]$	$[i^*-1, j-1]$	$[i, j^*]$	$[i, j-1]$	$[i, j-2]$
0	$[2, r-1]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\alpha_{i,j}$	$[2, i-1] \cup [j+1, r-1]$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Here $i > 1$, $j < r$, $i^* = max(i, 3)$, and $j^* = min(j, r-2)$.

Enumeration and Recurrences in Type A

Fibonacci-Like Recurrences

We write $\mathcal{A}_r(\lambda, \mu)$ for the Weyl alternation set in type A_r.

Theorem

Let $1 \leq i \leq j \leq r-2$ *and* $\mu = -\alpha_{i,j}$. *Then*

$$|\mathcal{A}_r(\tilde{\alpha}, \mu)| = |\mathcal{A}_{r-1}(\tilde{\alpha}, \mu)| + |\mathcal{A}_{r-2}(\tilde{\alpha}, \mu)|.$$

Idea: The first summand comprises elements of $\mathcal{A}_r(\tilde{\alpha}, \mu)$ that do not include a s_{r-1}. The second summand comprises elements that do.

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Theorem

Let $1 \leq i \leq j \leq r-2$ and $\mu = -\alpha_{i,j}$. Then

$$|\mathcal{A}_r(\tilde{\alpha}, \mu)| = |\mathcal{A}_{r-1}(\tilde{\alpha}, \mu)| + |\mathcal{A}_{r-2}(\tilde{\alpha}, \mu)|.$$

Idea: The first summand comprises elements of $\mathcal{A}_r(\tilde{\alpha}, \mu)$ that do not include a s_{r-1}. The second summand comprises elements that do.

Theorem

Let $h_r^i = |\mathcal{A}_r(\tilde{\alpha}, -\alpha_{i,r})|$. If $r \geq i+4$, then

$$h_r^i = h_{r-1}^i + h_{r-2}^i + 3h_{r-3}^i + h_{r-4}^i.$$

If $i, r \geq 3$, then

$$h_r^i = h_{r-1}^{i-1} + h_{r-2}^{i-2}.$$

Generating Function for Negative Roots

Theorem

Let $a_{r,i,j} = |\mathcal{A}_r(\tilde{\alpha}, -\alpha_{i,j})|$. *Then*

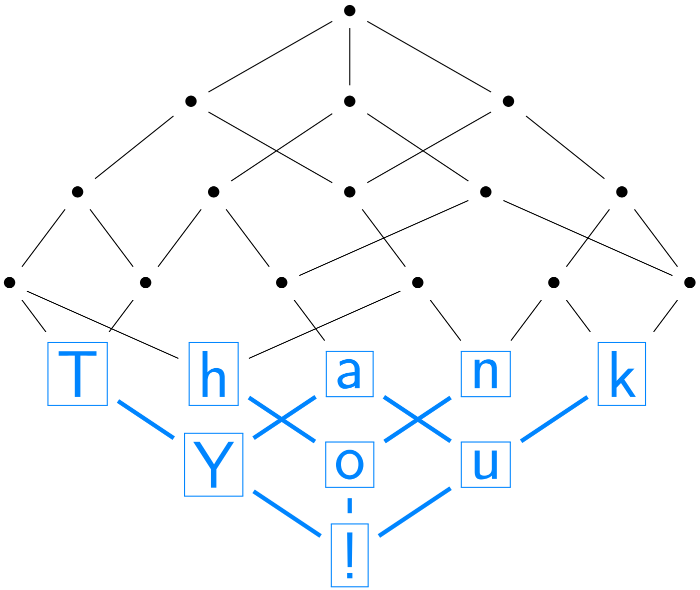
$$\sum_{1 \le i \le j \le r} a_{i,r,j} x^r s^i t^j = \frac{1}{t(1-x-x^2)} \left((1-x)t\mathcal{H}(xt,s) + \mathcal{P}(xt,s) - \frac{xst}{1-xst-(xst)^2} \right)$$

where

$$\mathcal{H}(x,s) = \frac{xs(x^5 s + 3x^4 s - xs + x^2 + 2x + 1)}{(1-x-x^2-3x^3-x^4)(1-xs-(xs)^2)}$$

and

$$\mathcal{P}(x,s) = \frac{xs(x^4 s + 3x^3 s + x + 1)}{(1-x-x^2-3x^3-x^4)(1-xs-(xs)^2)}.$$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